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SOFTWARE ALGORITHMS FOR LOW-COST STRAPDOWN INERTIAL NAVIGATION SYSTEMS OF SMALL UAV

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Abstract. This review involves the scope of problems, which arise in the developing of simple Strapdown Inertial Navigation Systems (SINS) designated for usage in small UAV and operating with low-cost inertial sensors. These problems include: elaboration of simple algorithms for rotational and translational state maintenance, developing of gyro-free accelerometer-based navigation systems (ASINS), correction of SINS errors on the basis of usage GPS, magnetometers and altimeters, and usage of sensors redundancy for developing fault-tolerant SINS and identification of the faulty sensor. Taking in account some peculiarities of small UAV flight missions (small operation radius of such UAV, restricted time of the flight mission and limited range of fight speeds), it was possible to develop algorithms for integration of differential equations for rotational and translational state maintenance, which allow producing of their solution in quadratures. Kalman filtering problem for GPS/SINS fusion was solved on the basis of the QR-factorization of covariance matrices. The algorithms are proposed for creation of ASINS and problems of relations between accelerometers redundancy, complexity of the ASINS algorithms and eventually the ASINS performance are considered.

Keywords: Strapdown Inertial Navigation System (SINS), accelerometer-based SINS, rate gyro, altimeter, magnetometer, Kalman filter, quaternion, sensor redundancy, fault detection.

AMS Subject Classification: 70-08, 93A30.

1. Introduction

Nowadays the deployment and applications of small low-cost UAV as well as the increasing of diversity of their flight missions are growing very intensively [8, 11, 18, 19]. These UAV must be equipped with low-cost navigation and flight control systems. That is why the research and development works in the area of small size and comparatively cheap inertial sensors for UAV strap-down inertial navigation systems (SINS) are intensively undertaken [9, 10, 18, 19, 55, 56], as well as in the area of development of the software algorithms for low-cost SINS [1, 18, 47]. If conditions of their application and peculiarities of their flight missions would be taken in consideration, then it is possible to simplify the traditional SINS algorithms [1, 4, 14, 16, 25, 33, 35 – 37, 44, 47, 57]. These peculiarities are based on the small operation radius of such UAV, restricted time of the flight mission and limited range of fight speeds. It gives possibility to neglect the Earth sphericity, the Earth rotation rate, etc. Meanwhile some other problems arise in the process of such systems design: the compensation of the sensors systematic errors, development of the SINS systems on the basis of accelerometers only without rate gyros, usage of the additional sources of information based on the readouts of magnetometers and altimeters, and the identification of sensors, which have failed during the SINS operation. It would be relevant to note that sometimes in such systems the problem of creating of the

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accelerometer-based SINS without usage of the rate gyros (RG) could be arisen especially in the cases of application of SINS in the fast spinning UAV[12, 58]. Taking in account aforementioned circumstances this review contains the algorithms of operation of simple SINS, which use RG and accelerometers, as well as the algorithms for SINS, which use accelerometers only. The problems of the external corrections of such systems using GPS [17, 26, 28, 29, 55], and other additional sources of information (magnetometer and altimeter) are also considered as well as the problems of the RG systematic errors compensation [47]. There are described the algorithms of the SINS errors correction, when several vectors with known positions in some basic frame are observed [45].

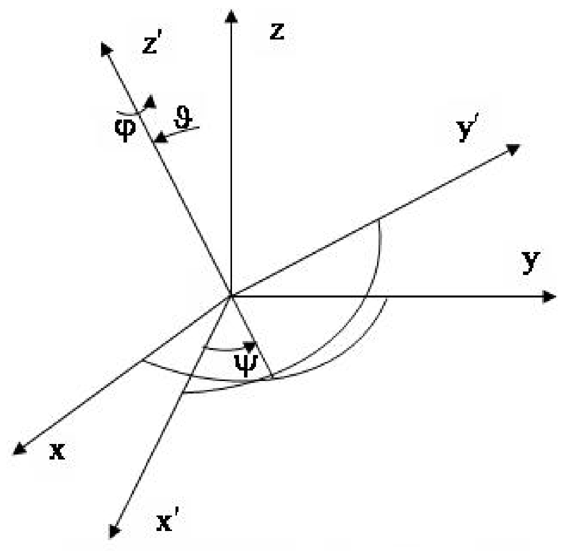

Figure 1. The fixed and body frames.

These algorithms also allow providing fault detection and the faulty sensor identification in a case of the sensors redundancy [49]. The motion control problems [2], the problems of the SINS sensors calibration [31, 32] as well as the initial alignment problems [60] are not considered in this review. Several results were obtained in accordance with the treaty of collaboration between the Institute of Mechanics named after S.P. Timoshenko of National Academy of Sciences of Ukraine and National Aviation University.

2. Basic relations for the rigid body attitude determination

From the very beginning it is expedient to present all known relations describing attitude determination and integrating of kinematics equations, which are the mathematical background for all results obtained in this review.We describe some different methods of attitude determination [13, 52, 54, 62]. In this review we will use the classical Euler angles ψ, ϑ, φ (precession, nutation and rotation) [1, 52], which determine the attitude of the rigid body, i.e. its transition from initial position in the fixed frame $Oxyz$ to the final position in the moving (body) frame $Ox'y'z'$ (see fig. 1). The choice of these angles is determined by the following considerations. As it is mentioned in [47, 52] the choice of Euler angles is very ambiguous in the different books and papers. So, in order to avoid this ambiguity, Euler angles, which are commonly used in the classical analytical mechanics, were chosen. Aforementioned transition could be performed via only one turn at the angle $\chi$with respect to the axis, having direction, which is determined by the anglesα, β, γ. That is why the attitude of body can be described by four Rodriguez-Hamilton parameters [52] $\lambda_0, \lambda_1, \lambda_2, \lambda_3$:

$$\lambda_1 = \cos\alpha \sin\frac{\chi}{2}, \lambda_2 = \cos\beta \sin\frac{\chi}{2}, \lambda_3 = \cos\gamma \sin\frac{\chi}{2}, \lambda_0 = \cos\frac{\chi}{2}.$$

It is obvious that: $\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$. Rodriguez-Hamilton parameters can be expressed via Euler angles as follows:

$$\lambda_0 = \cos\frac{\vartheta}{2}\cos\frac{\varphi+\psi}{2}, \lambda_1 = \sin\frac{\vartheta}{2}\cos\frac{\varphi-\psi}{2}, \lambda_2 = \sin\frac{\vartheta}{2}\sin\frac{\psi-\varphi}{2}, \lambda_3 = \cos\frac{\vartheta}{2}\sin\frac{\varphi+\psi}{2}. \quad (1)$$

The rigid body attitude with respect to the fixed frame $Oxyz$ can be determined by the direct cosine matrix (DCM) A for coordinates transform. If m is some vector in the fixed frame and components of vector k are the projection of this vector at the axes of the body frame $Ox'y'z'$, then:

$$k = Am. \quad (2)$$

This matrix can be expressed via Rodriguez-Hamilton parameters $\lambda_0, \lambda_1, \lambda_2, \lambda_3$:

$$A(\lambda) = \begin{bmatrix} \lambda_0^2+\lambda_1^2-\lambda_2^2-\lambda_3^2 & 2(\lambda_1\lambda_2+\lambda_0\lambda_3) & 2(\lambda_1\lambda_3-\lambda_0\lambda_2) \\ 2(\lambda_1\lambda_2-\lambda_0\lambda_3) & \lambda_0^2-\lambda_1^2+\lambda_2^2-\lambda_3^2 & 2(\lambda_2\lambda_3+\lambda_0\lambda_1) \\ 2(\lambda_1\lambda_3+\lambda_0\lambda_2) & 2(\lambda_2\lambda_3-\lambda_0\lambda_1) & \lambda_0^2-\lambda_1^2-\lambda_2^2+\lambda_3^2 \end{bmatrix} \quad (3)$$

The inverse relations have also take place. If, for instance, $A = [a_{ij}]$, $ij = \overline{1,3}$ and $1 + a_{11} + a_{22} + a_{33} > 0$, then due to [54, 62]: $\lambda_0 = \frac{1}{2}\sqrt{1 + a_{11} + a_{22} + a_{33}}$,

$$\lambda_1 = \frac{a_{23} - a_{32}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}}, \lambda_2 = \frac{a_{31} - a_{13}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}}, \lambda_3 = \frac{a_{12} - a_{21}}{2\sqrt{1 + a_{11} + a_{22} + a_{33}}} \quad (4)$$

It is expedient to present the expressions of the projections of the angular rate vector in the body frame $\omega_1, \omega_2, \omega_3$ via the Euler angles:

$$\omega_1 = \dot{\psi}\ \sin\vartheta\ \sin\varphi + \dot{\vartheta}\cos\varphi, \omega_2 = \dot{\psi}\ \sin\vartheta\ \cos\varphi - \dot{\vartheta}\sin\varphi, \omega_3 = \dot{\psi}\ \cos\vartheta + \dot{\varphi}. \quad (5)$$

If components of vector $\omega = [\omega_1\, \omega_2\, \omega_3]^T$ are measured and initial rigid body attitude is known, the vector of the Rodriguez-Hamilton parameters (quaternion) $\lambda = [\lambda_0\, \lambda_1\, \lambda_2\, \lambda_3]^T$ is determined as a result of the integration of the of kinematical equations (rotational state maintenance [1, 14, 41, 54]):

$$\dot{\lambda} = \frac{1}{2}\cdot \Omega\lambda, \quad (6)$$

$$\Omega = \begin{bmatrix} 0 & -\omega_1 & -\omega_2 & -\omega_3 \\ \omega_1 & 0 & \omega_3 & -\omega_2 \\ \omega_2 & -\omega_3 & 0 & \omega_1 \\ \omega_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}, \|\lambda\|^2 = \lambda^T\lambda = 1.$$

Hereafter $\|\cdot\|$ denotes the spectral matrix norm; upper index T stands for the matrix transposition.

If Euler angles are small and trihedrons $Oxyz$ and $Ox'y'z'$ are very close to each other, then it is possible to use approximate expression for DCM A_s based on relations (26) in [62] and (8.2) in [41]:

$$A_s \cong \begin{bmatrix} 1 & \mu_3 & -\mu_2 \\ -\mu_3 & 1 & \mu_1 \\ \mu_2 & -\mu_1 & 1 \end{bmatrix}, \quad (7)$$

where μ_1, μ_2, μ_3 are small angles of turn of the trihedron $Oxyz$ with respect to the axes x, y, z respectively.

3. Algorithms of the SINS software

As it is known [25, 28, 55, 56] the algorithms of the SINS software consist of the algorithms of the rotational and translational state maintenance. We begin from the rotational state maintenance.

The procedure of the equation (6) numerical integration can be represented as follows. As far as this procedure is discrete, small sampling period δt is used. The basic idea of this procedure consists of its solution determination in quadratures instead of its solving in real time. In order to achieve this result in [41] the quadratic spline approximation of the angular rate vector was proposed. . The readouts of rate gyros in the discrete moments of time $t_{i-2},\ t_{i-1},\ t_i$ are known: $\omega(t_{i-2})$, $\omega(t_{i-1})$,$\omega(t_i)$. Using these values and quadratic spline approximation, it is possible to determine quasi-coordinates (the components of the vector $\nabla\theta_i = \int_{t_i}^{t_i+\delta t}\omega dt$) in the following way:

$$\nabla\theta_i = \frac{\Delta t}{12}(5\omega(t_i) + 8\omega(t_{i-1}) - \omega(t_{i-2})) \tag{8}$$

Then we can express solution of equation (6) $\delta\lambda(t_i)$ at the time period δt with initial condition $\begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$ (calculating quaternion, which corresponds to the small turn of rigid body during time δt) as function of $\nabla\theta_i$. After this the rigid body attitude determination is calculated by consequent multiplication $\delta\lambda(t_i)$ (“elementary” quaternion) by the value of quaternion in the previous moment of time $\lambda(t_{i-1})$:

$$\lambda(t_i) = \lambda(t_{i-1})\delta\lambda(t_i). \tag{9}$$

Matrix representation of this procedure looks like follows:

$$\lambda(t_i) = \begin{bmatrix} \delta\lambda_0(t_i) & -\delta\lambda_1(t_i) & -\delta\lambda_2(t_i) & -\delta\lambda_3(t_i) \\ \delta\lambda_1(t_i) & \delta\lambda_0(t_i) & \delta\lambda_3(t_i) & -\delta\lambda_2(t_i) \\ \delta\lambda_2(t_i) & -\delta\lambda_3(t_i) & \delta\lambda_0(t_i) & \delta\lambda_1(t_i) \\ \delta\lambda_3(t_i) & \delta\lambda_2(t_i) & -\delta\lambda_1(t_i) & \delta\lambda_0(t_i) \end{bmatrix} \begin{bmatrix} \lambda_0(t_{i-1}) \\ \lambda_1(t_{i-1}) \\ \lambda_2(t_{i-1}) \\ \lambda_3(t_{i-1}) \end{bmatrix} \tag{10}$$

$$\delta\lambda(t_i) = \begin{bmatrix} \delta\lambda_0(t_i) & \delta\lambda_1(t_i) & \delta\lambda_2(t_i) & \delta\lambda_3(t_i) \end{bmatrix}^T.$$

In [13] the expressions of quaternions $\delta\lambda(t_i)$ as functions of the quasi-coordinate vector $\nabla\theta_i$, which guarantee this or those approximation accuracy of quaternion $\delta\lambda(t_i)$ depending on these expressions complexity. As it was proved in [7, 43] this complexity could be diminished from the point of view of the following circumstance. Let rigid body is slightly turned with respect to the x-axis at the angle $\chi = \int_0^{\delta t}\omega_0 dt$. Let the estimated value of this small turn will be φ. It was shown in [7, 43], that from the point of view of attitude determination accuracy it is necessary to minimize error $|\varphi - \chi|$ instead of minimization of vector λ approximation error. On the basis of this result in [43] the following expression of vector $\delta\lambda^2$, which provides the order of the approximation error $O(\chi^2)$, was proposed

$$\delta\lambda_m^2 = \begin{bmatrix} 1 - 1/12\,\|\nabla\theta_i^*\|^2 \\ 1/2\nabla\theta_i^* \end{bmatrix}.$$

In [42] the expression of vector $\delta\lambda^3$ (order of the approximation error $O(\chi^3)$) was developed:

$$\delta\lambda_m^3 = \begin{bmatrix} 1 - 1/12\,\|\nabla\theta_i^*\|^2 \\ 1/2\nabla\theta_i^* - 1/24\left(\nabla\theta_i^* \times \nabla\theta_{i-1}^*\right) \end{bmatrix}. \tag{11}$$

It must be noted about existing of the general relations, which allow to evaluate the accuracy of the kinematics equations (6) integration. However these relations are very complicated. Therefore it is necessary to mark that in [7] simple formulas were developed in the case of the finite motion for the estimation of the average rate of the drift, when this or those integration algorithm is applied. Besides this, some notion about the accuracy of the equation (6) integration for various algorithms of the $\delta\lambda$ construction gives examples considered in [7].

It is necessary to note that in the cases, which are given below the expression (11) is applied. Expressions (8) – (11) are the mathematical background of the rotational maintenance software.

Describing the integration of the translational motion kinematics equations it is necessary to note that these equations are based on the Coriolis theorem [14, 41]:

$$\frac{dv}{dt} = w - 2\Omega_z \times v - \Omega_z \times \Omega_z \times R \tag{12}$$

where w stands for absolute acceleration, v is the relative UAV speed, Ω_z is the Earth angular rate, R is the radius-vector of UAV mass point in the geometric frame. Output signals of accelerometers (w_a) are determined with the following relation:

$$w_a = w + g, \tag{13}$$

where g is the gravity accelerations. Taking into account the simplifying assumptions mentioned in the Introduction, which are valid for small UAV, it is possible to neglect the Coriolis acceleration for these cases [14], i.e. to exclude term $2\Omega_z \times v$ from the right part of (12). However, if the more powerful on-board computers and the more precise sensors are available, then it would be possible to include Coriolis acceleration in the equations of the UAV motion.

In order to map acceleration vector in the navigation frame, corresponding to equation (12) it is necessary to integrate equation (10) using given initial conditions, defined by initial alignment procedure [60], and expression (11) as elementary quaternion $\delta\lambda$. After obtaining current value of quaternion $\lambda(t_i)$ as a result of this integration, we can determine DCM A using expression (3). It gives possibility to map acceleration vector, which is defined by accelerometers readouts, from body frame to navigation frame in accordance with (2), thus determining $\tilde{w}$ in the right part of the equation (12). Having the values $\tilde{w}$ in the sampling moments $t_{i-2},\ t_{i-1},\ t_i$, we can write expressions for $v(\,t_i)$, $r(\,t_i)$, which are similar to the expression (8):

$$v(\,t_i) = (5\tilde{w}(t_i) + 8\tilde{w}(t_{i-1}) - \tilde{w}(t_{i-2})) \cdot \frac{\Delta t}{12} + v(t_{i-1}), \tag{14}$$

$$r(\,t_i) = (3\tilde{w}(t_i) + 10\tilde{w}(t_{i-1}) - \tilde{w}(t_{i-2})) \cdot \frac{\Delta t^2}{24} + v(t_{i-1}) \cdot \Delta t + r(t_{i-1}). \tag{15}$$

Summarizing aforementioned, it is possible to say, that the expressions (2), (3), (8) – (15) determine the algorithm of the SINS operation, i.e. they allow to produce the estimation of the navigation parameters in the sampling moments t_i on the basis of the RG and the accelerometers readouts in the moments of time t_i. Proposed algorithm of the SINS operation doesn't contain the integration of the kinematics equations and uses of final results of this integration in quadratures. This, in turn, allows using traditional algorithms of the GPS and the SINS fusion for correction of the considered SINS.

4. Model of sins without usage of rate Gyros

As it was noted in the introduction this kind of SINS are very useful in a case of its application in the fast spinning UAV, when, as it was underlined in [9, 12], the RG application might be very problematic. Similar to [46] we consider two problems of determination of the rigid body kinematics parameters on the basis of the accelerometers' measurements only. In the first problem the angular rate vector and velocity of point, which is assumed as the origin of the moving body frame, are determined on the basis of measurement of three body points' velocities. In the second problem the angular and the linear accelerations of the moving frame are calculated on the basis of observations of three body points' accelerations and known angular rate of rigid body. Then on the basis of the achieved results the model of the accelerometer-based SINS (ASINS) without RG usage is investigated.

4.1 Determination of velocities The problem of determination of the rigid body angular rate vector and velocity vector of one of its point on the basis of observation of its three points velocities was considered by several authors (see [22, 50], where further references were cited). The problem is formulated in the following way (see Fig.2).

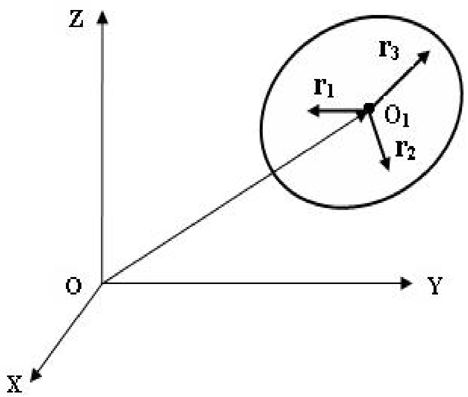

Figure 1. Allocation of the velocity sensors.

Three vectors r_1, r_2, r_3 define points, where linear velocity is measured. Using results of these measurements it is necessary to determine the vector of the body's angular rate ($\omega = [\omega_1\, \omega_2\, \omega_3]^T$) and the linear velocity vector ($v_0 = [v_1\, v_2\, v_3]^T$) of the body frame origin 1. Taking into account known expression (see, for instance, (2.7.8) [52], and (2) [58]), which defines the rigid body's velocity depending on vector r:

$$v = v_0 + \omega \times r, \tag{16}$$

it is possible to write the following linear expressions (equations (6) [50], (4) [22]) uniting the searched components of the ω, v vectors and the results of the three points' velocities observations:

$$V = \Omega P + v_0 h^T, \tag{17}$$

where: $\Omega = \omega\times = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}$, $P = [r_1\ r_2\ r_3]$, $h = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$, and V is the matrix, the columns of which are the vectors of three points velocities, defined by vectors r_1, r_2, r_3.

Let $\beta_1, \beta_2, \beta_3$ and $\gamma_1, \gamma_2, \gamma_3$ are the columns of the P^T and V^T matrices:

$$P^T = [\beta_1, \beta_2, \beta_3]\,, V^T = [\gamma_1, \gamma_2, \gamma_3]\,.$$

In this case, expression (17) can be written as the system of linear equations with respect to ω, v_0:

$$A_v x = B, \tag{18}$$

$$x = \begin{bmatrix} \omega \\ v_0 \end{bmatrix}, A_v = \begin{bmatrix} o & \beta_3 & \beta_2 & h & o & o \\ -\beta_3 & o & \beta_1 & o & h & o \\ \beta_2 & -\beta_1 & o & o & o & h \end{bmatrix}, B = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix},$$

where o is the (3×1) zero matrix.

Taking into account that velocity measurements are corrupted with noises, we rewrite (18) in the following form:

$$A_v x = B_0 + n_v, \tag{19}$$

where n_v is the measurement errors vector and vector B_0 is formed from true values of velocities of considered points.

4.2. Determination of accelerations

Likewise to the previous problem it is possible to consider the problem of determination of the body's angular acceleration vector and the linear acceleration of one of its points based on the results of the body's three points' accelerations observation. Let three vectors ρ_1, ρ_2, ρ_3 define three points of the rigid body. In each of these points three accelerometers are located, which allow to record acceleration vector components in the given points. Using the results of these measurements and the angular rate vector value ($\omega = [\omega_1\, \omega_2\, \omega_3]^T$) it is necessary to determine the angular acceleration vector $\left(\varepsilon = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3]^T = \frac{d\omega}{dt}\right)$ and the linear acceleration vector ($w_0 = [w_1\, w_2\, w_3]^T$) of the body frame origin. Concerning considered problem, the expression (16) might be considered as an analog of the expression ((2.17.9) [52]), which calculates the acceleration (w) of the rigid body's point determined by vector ρ:

$$w = w_0 + \varepsilon \times \rho + \omega \times (\omega \times \rho). \tag{20}$$

Denoting $U = [W_1\, W_2\, W_3]$, where W_i are acceleration vectors of the points, which are determined by ρ_i $(i = 1, 2, 3)$, it is possible on the basis of (20) to write the analog of the expression (17):

$$U = \Omega^2 P_w + EP_w + w_0 h^T. \tag{21}$$

Here: $P_w = [\rho_1\ \rho_2\ \rho_3]$, $E = \varepsilon\times = \begin{bmatrix} 0 & -\varepsilon_3 & \varepsilon_2 \\ \varepsilon_3 & 0 & -\varepsilon_1 \\ -\varepsilon_2 & \varepsilon_1 & 0 \end{bmatrix}$, and matrices Ω, h are similar to the matrices in the expression (17).

Likewise to (17) it is possible to represent expression (21) as the system of linear equations with respect to ε, w_0. Let $\alpha_1, \alpha_2, \alpha_3$; $\delta_1, \delta_2, \delta_3$; $\sigma_1, \sigma_2, \sigma_3$ are the columns of the matrices $U^T, P_w^T,\ (\Omega^2 P_w)^T$, i.e.

$$U^T = [\alpha_1\ \alpha_2\ \alpha_3]\,, P_w^T = [\delta_1\, \delta_2\, \delta_3]\,, (\Omega^2 P_w)^T = [\sigma_1\, \sigma_2\, \sigma_3]\,.$$

Then it is possible to write expression (21) in the form, which is similar to (18), namely

$$A_w x = B_\omega + B_w, \tag{22}$$

$$x = \begin{bmatrix} \varepsilon \\ w_0 \end{bmatrix}, A_w = \begin{bmatrix} o & \delta_3 & \delta_2 & h & o & o \\ -\delta_3 & o & \delta_1 & o & h & o \\ \delta_2 & -\delta_1 & o & o & o & h \end{bmatrix}, B_\omega = \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix}, B_w = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix},$$

where o is the 3×1-size zero matrix likewise to (18).

As well as in the case of the expression (10), we suggest that accelerometers outputs are corrupted with noises. Then (22) could be rewritten in the following form:

$$A_w x = B_\omega + B_{wo} + n_w, \tag{23}$$

where n_w are acceleration errors and components of vector B_{wo} are formed by true values of accelerations.

4.3 Accelerometer-based SINS

Taking into account that $w_0 = \frac{dv_0}{dt}$, $\varepsilon = \frac{d\omega}{dt}$, system (23) could be considered as system of nonlinear differential equations with respect to ω. In other words, considering accelerometers signals (B_w) as known external influences, it is possible to find $\omega(t)$ and $v_0(t)$ by integration the system (23) with known initial conditions. So considered approach allows acquiring information about vehicle's angular rate without usage of the rate gyros. However in this case it is necessary to take into account the following circumstances, when such SINS would be designed.

Vector x appearing in (23) is given in the moving frame. So far as we are interesting in the position of vehicle in the inertial frame, it is expedient to map the second component of the vector x (vector w_0) in the inertial frame and to perform further integration in it, which would allow to determine in the moving frame velocity and the coordinates of the vehicle's point, which is accepted as the moving frame's origin. Meanwhile the first part of the vector x (vector ω) might be used for the determination of the current body's orientation, which could be determined by Rodriguez-Hamilton parameters (1), as well as by direct cosine matrix (2) (relation between them is determined by expressions (3), (4)). In our case it is convenient to determine Rodriguez-Hamilton parameters by integration of equation (6), in which vector ω components can be found in process of the equation (23) integration. Then it is possible to define in accordance with (3) numerical values of the matrix A entries, which are used for the mapping of w_0 into the inertial frame. This matrix allows mapping vector w_0 in the inertial frame and, as it was noticed before, to determine the current values of the velocity and vehicle's coordinates by integration. So the implementation of such kind of inertial system includes:

- finding ω by integration of 3 differential equations (the first three equations in (23));
- finding quaternionλ, which determines in accordance with (3) DCM A (which will allow to map

acceleration vector w_0 in inertial frame) by integration of system (6) (4 equations);
- determination of the vehicle's velocity and coordinates by integration of 6 equations.

In other words it is necessary to integrate the 13-th order system of the differential equations. Initial conditions for this system would be the values in the initial moment of time of the following variables: vehicle's coordinates ($r_0 = [x_0\ y_0\ z_0]^T$), initial attitude (quaternion $\bar{\lambda}$) or corresponding DCM $A(\bar{\lambda})$, vehicle's velocity vector ($\bar{v}_0 = [v_{x0}\ v_{y0}\ v_{z0}]^T$), vehicle's angular rate vector ($\omega_0 = [\omega_{x0}\ \omega_{y0}\ \omega_{z0}]$).

Note that it is possible to determine $\bar{v}_0$, ω_0 by results of measurements with GPS the velocities of the three body's points using algorithm described in p. 3.1. For this purpose it is necessary to use GPS having possibility of attitude determination [30].

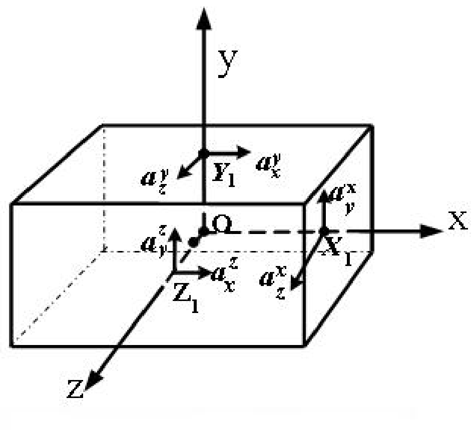

Figure 1. The scheme of allocation of accelerometers in the rigid body.

So far as the size of matrix A_w in (23) is equal 9×6, then it is possible to eliminate from consideration in system (23) three rows (to exclude the readouts of three accelerometers). In other words, it is possible to create the inertial navigation system using only 6 accelerometers (see [46, 48, 58]). For illustration of this statement let us consider the scheme of allocation of accelerometers depicted in the Figure 3. Here X_1, Y_1, Z_1 are points of the axes OX, OY, OZ, where couples of accelerometers are installed. The orientation of their sensitivity axes is represented at this Figure. For example, a_x^y denotes, that this accelerometer measures the acceleration of the point Y_1 in the direction of the axis OX. Using this scheme of allocation of accelerometers, it is possible in the system (15) of 9 equations to remain 6 equations, deleting the 1st, 5th and the 9th rows. Thus, if the angular velocity vector B_ω is known, then it is sufficient to install 6 accelerometers in order to determine vectors ε and w_0. Note that in [55] another scheme of the accelerometers allocation is given, which allows determining directly the angular acceleration vector ε as a linear combination of the accelerometers readouts. It is useful to note that such inertial system using comparatively simple sensors might not provide acceptable accuracy of navigation parameters determination especially during significant operation time. In this situation it is expedient to augment this system by GPS [28, 30, 47, 55].

It is obvious, that the accuracy of the determining of the current value of the angular rate vector as a result of integration of the angular acceleration will significantly depend on the accuracy of definition of the initial value of the angular rate vector in the initial moment of time. For decreasing this dependence it is expedient to increase the amount of accelerometers and to use obtained redundant information for increasing the accuracy of estimation of the current value ω.

4.4. Discrete version of ASINS.

It was noted in previous item 3.3 that the ASINS operation is based on solving of system of differential equations having 13th order. From the point of view of its computer implementation it is expedient to consider the discrete version of ASINS, i.e. to consider the case, when sensors' readouts can be acquired in the sampling times with sampling period Δt with sampling frequency $f = {}^1\!/_{\Delta t}$. Therefore the sought-for navigation parameters (DCM $A(\lambda)$, velocity v,and coordinates' radius-vector r) will be calculated after sampling period. It was shown using certain example in [7], that it possible to use simple approximation (Euler's method) for calculation of quaternion $\delta\lambda\,(t_i)$ and quasi-coordinate $\nabla\theta_i$:

$$\delta\lambda\,(t_i) = \begin{bmatrix} 1 \\ {}^1\!/_2\Delta\theta_i \end{bmatrix}, \nabla\theta_i = \frac{\omega\,(t_i) + \omega\,(t_{i-1})}{2}\,\Delta t, \tag{24}$$

$$\omega\,(t_i) = \omega\,(t_{i-1}) + \varepsilon\,(t_i)\,\Delta t,$$

where $\varepsilon\,(t_i)$ stands for the angular acceleration vector, which are determined with expression (22) on the basis of the accelerometers' readouts in the sampling moment t_i , taking in account that in (21) the components of the vector $\omega\,(t_{i-1})$ are used as the entries of the matrix Ω. Having the estimation (24) of the quaternion $\delta\lambda\,(t_i)$, we can find quaternion $\lambda\,(t_i)$ using (9), (10), and then in accordance with (3) to find matrix $A\,(\lambda\,(t_i))$. Using matrix $A\,(\lambda\,(t_i))$ determined by expression (22) for mapping vector $w_0\,(t_i)$ into the fixed frame and taking into account expressions (12), (13), it is possible to find estimates of velocity $(v\,(t_i))$ and coordinates $(r\,(t_i))$ of UAV.

$$v\,(t_i) = v\,(t_{i-1}) + \tilde{w}\,(t_i)\,\Delta t,$$

$$r\,(t_i) = r\,(t_{i-1}) + \frac{v\,(t_i) + v\,(t_{i-1})}{2}\Delta t.$$

These expressions are the analogs of expressions (14), (15).

So relations given above allow finding the estimates of the navigation parameters in the sampling moments t_i using the accelerometers' readouts in these moments. That is why this

SINS can be the subsystem of the integrated GPS/INS navigation system for comparatively simple objects likewise to those, which were considered, for example, in [46], [48].

4.5. Increasing of the accuracy of the angular rate vector estimation.

Here it is demonstrated that degree of redundancy of the accelerometers can be used for increasing of the accuracy of the angular rate vector estimation [48]. In the previous item 3.3 the case of 6 accelerometers usage was considered. Now we will consider the case of 9 accelerometers usage.

Let system depicted at the Figure 3 will be augmented with 3 accelerometers located in the point O. Their sensitivity axes are directed along the axes OX, OY, OZ respectively, so these accelerometers are measuring the components of acceleration vector of origin. The readouts of these accelerometers are denoted as a_x^0, a_y^0, a_z^0. Consider that the distance from the origin of each point X_1, Y_1, Z_1 is equal L. Denote also: $n_y^x = a_y^x - a_y^0$, $n_x^y = a_x^y - a_x^0$, $n_z^x = a_z^x - a_z^0$, $n_z^y = a_z^y - a_y^0$, $n_x^z = a_x^z - a_x^0$, $n_y^z = a_y^z - a_y^0$. If this scheme of the accelerometers allocation is used, it is possible to derive from (20) or (21) the following equations:

$$2L\varepsilon_1 = n_z^y - n_y^z, 2L\varepsilon_2 = n_x^z - n_z^x, 2L\varepsilon_3 = n_y^x - n_x^y; \tag{25}$$

$$2L\omega_2\omega_3 = n_z^y + n_y^z, 2L\omega_1\omega_3 = n_x^z + n_z^x, 2L\omega_1\omega_2 = n_y^x + n_x^y. \tag{26}$$

Note, that equations (25) coincide with equations (3.390) in [5]. So in a case of 9 accelerometers equations (25) determine additional 3 variables: $\omega_1\omega_2$, $\omega_1\omega_3$, $\omega_2\omega_3$. It is expedient to use this information for correction of the results of angular acceleration ε integration. Note also, that if 2 from 3 components of vector ω are equal to zero (the rotation takes place with respect to only one fixed axis), the relations (25) can't be used for correction the results of integration. From this point of view it is expedient to augment the 9-accelerometer measuring system, which is described above, with additional three accelerometers.

Now we will consider the case of 12 accelerometers usage. Additional 3 accelerometers are located in the following way: in the point X_1 the acceleration along axis OX is now measured, as well as in the points Y_1 and Z_1 the accelerations along axes OY and OZ respectively are measured also (see Figure 3). Note that scheme of allocation of accelerometers coincides with the scheme, shown at the Figure 3.7 in [5]. Let the readouts of these accelerometers are equal to a_x^x, a_y^y, a_z^z. Denote: $n_x^x = a_x^x - a_x^0$, $n_y^y = a_y^y - a_y^0$, $n_z^z = a_z^z - a_z^0$. For this 12-accelerometer measuring system relations (25), (26) must be augmented as follows:

$$2L\omega_1^2 = n_x^x - n_y^y - n_z^z, 2L\omega_2^2 = -n_x^x + n_y^y - n_z^z, 2L\omega_3^2 = -n_x^x - n_y^y + n_z^z. \tag{27}$$

So in the considered case of 12 accelerometers the relations (26), (27) could be used for correction of results of integration.

Consider briefly the problem of the relations (26) usage for increasing the accuracy of the vector ω estimation in cases of 9 and 12 accelerometers usage. In the last case besides aforementioned relations (26) additional relations (27) can be used. Therefore consider 9-accelerometer case, which readouts determine the angular acceleration vector ε from expressions (25), as well as vector $\Omega_n = [\omega_2\omega_3 \;\; \omega_1\omega_3 \;\; \omega_2\omega_1]^T$ from expressions (26). Let current value ω be determined by expression (24), which can be rewritten as follows:

$$\omega(t_1) = \omega(t_{i-1}) + \Delta\omega_i, \Delta\omega_i = \frac{\varepsilon(t_i) + \varepsilon(t_{i-1})}{2}\Delta t.$$

So the estimation of the vector $\omega(t_{i-1})$ increment, obtained as a result of calculation of the angular acceleration vector ε, is determined by relation $\Delta\bar{\omega}_i = \frac{\varepsilon(t_i)+\varepsilon(t_{i-1})}{2}\Delta t$. From the other hand, considering $\Delta\bar{\omega}_i$ as a small value, it is possible to write the following relations:

$$\Omega_n = H\Delta\omega_i + \Omega_{n0}, H = \begin{bmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & \omega_1 \\ \omega_2 & \omega_1 & 0 \end{bmatrix}, \Omega_{n0} = \left[\omega_2\omega_3 \;\; \omega_1\omega_3 \;\; \omega_1\omega_2\right]^T. \tag{28}$$

Components of vector Ω_n in (28) are determined by (26), and the components of vector ω, which determine H and Ω_{n0}, correspond to the values of vector $\omega(t_{i-1})$ components. In other words, as a result of previously made assumption about small value of $\Delta\omega_i$, we have the standard problem of the parameter estimation by weighted least square method [15]. In accordance with (28) the following vector z is observed:

$$z = \Omega_n - \Omega_{n0} = H\Delta\omega + \nu, \tag{29}$$

where ν is the vector of the measurement errors. The estimation of the $\Delta\,\widehat{\omega}_i$ value is determined by relation (12,2, 7) from [33]:

$$\Delta\,\widehat{\omega}_i = \Delta\bar{\omega}_i + PH^TR^{-1}(z - H\Delta\bar{\omega}_i), P^{-1} = M^{-1} + H^TR^{-1}H. \tag{30}$$

Here M is the covariance matrix of the estimation errors of $\Delta\,\widehat{\omega}_i$ and R is covariance matrix of measurement errors ν in (29). Finally, the value of vector ω in the moment t_i is determined by relation:

$$\omega(t_i) = \omega(t_{i-1}) + \Delta\,\widehat{\omega}_i, \tag{31}$$

where $\Delta\,\widehat{\omega}_i$ can be found from (13). Note, that matrix P^{-1} might be ill-conditioned. Then for determination of matrix P in (30) it could be expedient to use the approach, described in [36, 45].

As far as the matrices M, R are symmetric and positively defined, they can be represented in the following form: $M = m^2,\ R = r^2$, or $m = M^{1/2},\ r = R^{1/2}$. Correspondingly it is possible to represent the expressions for matrix P^{-1} as follows [24, 51, and 61]:

$$P^{-1} = \left[m^{-1} \;\; H^Tr^{-1}\right] \left[m^{-1} \;\; H^Tr^{-1}\right]^T. \tag{32}$$

Using QR-factorization procedure, we can transform matrix $\left[m^{-1} \;\; H^Tr^{-1}\right]^T$ to the following form:

$$\left[m^{-1} \;\; H^Tr^{-1}\right]^T = Q\left[\rho \; 0\right]^T, \tag{33}$$

where Q is orthogonal matrix, and ρ is the invertible matrix. Taking into account, that $Q^TQ = I$ and substituting (33) in (32), we obtain: $P^{-1} = \rho^T\rho$, or $P = \rho^{-1}\rho^{-T}$.

Thus the 1st expression (30) can be represented in the following form:

$$\Delta\,\widehat{\omega}_i = \Delta\bar{\omega}_i + \rho^{-1}\rho^{-T}H^TR^{-1}(z - H\Delta\bar{\omega}_i). \tag{34}$$

If we suppose, that $M = \mu^2I$, $R = \gamma^2I$, then the relation (34) can be written as:

$$\Delta\,\widehat{\omega}_i = \Delta\bar{\omega}_i + \rho^{-1}\rho^{-T}H^T(z - H\Delta\bar{\omega}_i), \tag{35}$$

where ρ is determined by QR-factorization of the following matrix:

$$\left[\lambda I \;\; H^T\right]^T, \lambda = \frac{\gamma}{\mu}. \tag{36}$$

Note, that described algorithm of correction can be used in a case of 12 accelerometers. In this case the matrix H and the vector Ω_{n0} in (28) have the following forms:

$$H = \begin{bmatrix} 0 & \omega_3 & \omega_2 & 2\omega_1 & 0 & 0 \\ \omega_3 & 0 & \omega_1 & 0 & 2\omega_2 & 0 \\ \omega_2 & \omega_1 & 0 & 0 & 0 & 2\omega_3 \end{bmatrix}^T, \Omega_{n0} = \begin{bmatrix} \omega_2\omega_3 & \omega_1\omega_3 & \omega_1\omega_2 & \omega_1^2 & \omega_2^2 & \omega_3^2 \end{bmatrix}^T.$$

Here, likewise to the 9 accelerometers case, entries of H and Ω_{n0} are determined by components of vector $\omega(t_{i-1})$. Components of vectorΩ_n are determined by relations (26), (27).

5. External correction of the sins errors

In this item the algorithms of the SINS error correction on the basis of GPS data are considered. The correction algorithms based on the combined usage the data of GPS, magnetometer and altimeter are described also. This problem is considered via linear approximation, i.e. in the framework of the Kalman filter.

5.1. SINS errors correction via GPS [38-41, 45, 47]

Let μ, δv, δr are the vectors of the SINS errors in the frame, which is used in the equation (8), namely: μ is the vector of the small turn of the attitude determination error, which defines DCM A_s in (7); δv, δr are the vectors the errors of determination of the velocity and the coordinates of the moving vehicle. Let $\delta c$is the vector of the RG systematic errors and $w = [w_1, w_2, w_3]^T$ is the vector of the full acceleration. All aforementioned errors are components of the Kalman filter state vector $x = [\mu \ \delta v \ \ \delta r \ \ \delta c]^T$; $x \in R^{12\times 1}$. The direct cosine matrix (DCM) A is defined with expression (3). The equation of the SINS errors propagation is accepted in the form, which is similar to the equation (7.149) [27]:

$$\dot{x} = Fx + n, \tag{37}$$

$$x = \begin{bmatrix} \mu \\ \delta v \\ \delta r \\ \delta c \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & 0 & A^T \\ C & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, C = \begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix},$$

where n is the white noise vector. Hereafter 0 is the zero matrix of the corresponding size, and I is the unit matrix of the corresponding size. The following equation, which is the discrete analog of the equation (37), i.e. the equation of the errors propagation in the sampling moments t_k, is accepted as:

$$x_{k+1} = \Phi_k x_k + n_k, \tag{38}$$

$$\Phi_k = I + F\Delta t + \frac{(\Delta t)^2}{2}F^2 = \begin{bmatrix} I & 0 & 0 & A^T\Delta t \\ C\Delta t & I & 0 & CA^T\frac{(\Delta t)^2}{2} \\ C\frac{(\Delta t)^2}{2} & I\Delta t & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

where n_k stands for the vector of the random errors of the SINS operation. Subscript k indicates corresponding sampling moment $k\Delta t$. Let us assume that information about estimation of the UAV coordinates and velocity is produced by the SINS and GPS operation at the $k-th$ sampling moment, i.e. the following observation process takes place:

$$z_k = Hx_k + \xi_k, \tag{39}$$

$$H = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{bmatrix},$$

where ξ_k is the measurement error.

Thereby, using relations (39), the problem of the SINS correction can be formulated as the optimal filtration problem. It is known (see, for instance, item 12.4 in [15]), that solution of this problem can be found in the following form:

$$\begin{aligned} \hat{x}_k &= \bar{x}_k + K_k \left(z_k - H\bar{x}_k \right), \\ \bar{x}_{k+1} &= \Phi_k \hat{x}_k. \end{aligned} \tag{40}$$

The filter gain matrix (K_k), which generates the vector of the optimal estimation, is determined in the following way:

$$K_k = M_k H^T \left(H M_k H^T + R_k \right)^{-1}. \tag{41}$$

$$M_{k+1} = \Phi_k S_k \Phi'_k + Q_k^T, \tag{42}$$

$$S_k = M_k - K_k \left(H M_k H^T + R_k \right) K_k^T. \tag{43}$$

Here Q_k, R_k are covariance matrices of noises n_k,ξ_k, which are present in the expressions (38), (39). Matrix M_0 is the covariance matrix of the vector x initial estimation, which is assumed to be given. Note that correction of the SINS errors is done after $j > 1$ sampling periods. In this case the variation of the SINS errors is done in correspondence with equation (38), meanwhile the variations of their covariance matrices is done in correspondence with (42) (it is possible to assume that at these sampling periods $H = 0$). At the sampling period k, when correction of the covariance matrix is performed, this correction is described by equation (43).

Thereby the first 9 components of vector x_k in (40) determine the estimations of the error vectors $\mu_k, \delta v_k, \delta r_k$, and consequently the estimations of the attitude determination, velocity and UAV coordinates in the sampling period t_k. Last three components of the vector x_k (or vector δc_k) define the estimations of the systematic errors of the RG readouts. It is expedient to use these estimations for correction of the RG readouts. Thus, if in the moment t_k the RG readouts are the vector $\bar{\omega}(t_k)$ components, then in the relation (8) it is necessary to use the following value of the angular rate vector:

$$\omega(t_k) = \bar{\omega}(t_k) - \delta \overset{\frown}{c}, \delta \overset{\frown}{c} = \sum_k \delta c_k. \tag{44}$$

Underline that correction δc_k changes its value only in the moment of correction SINS via GPS data, i.e. in the moments, when $H \neq 0$.

Essential peculiarity of this problem is the property of incomplete observability of the pair of Φ_k and H matrices [38]. This circumstance requires increasing of the computational procedures accuracy. For this kind of problems there are commonly used algorithms, which allow computing the Cholesky's factors for corresponding matrices. Below the algorithm [38] based on the QR factorization of these matrices will be described. It is assumed, that the matrix R_k is convertible. The general case is described in [35]. Let m_k, p_k, q_k, η_k stand for the Cholesky's factors of the M_k, S_k, Q_k, R_k matrices respectively, i.e.

$$M_k = m_k m_k^T, \quad \mathrm{S}_k = p_k p_k^T, \quad \mathrm{Q}_k = q_k q_k^T, \quad \mathrm{R}_k = \eta_k \eta_k^T.$$

In the case of the matrix R_k convertibility the relation (43) can be rewritten in the following form:

$$p_k p_k^T = m_k \left(I + m_k^T H^T R_k^{-1} H m_k \right)^{-1} m_k^T. \tag{45}$$

We represent the expression in the brackets in the form of the product of two rectangular matrices:

$$\begin{aligned} &I + m_k^T H^T R_k^{-1} H m_k = N_k N_k^T \\ &N_k = \begin{bmatrix} I & m_k^T H^T \eta_k^{-1} \end{bmatrix}. \end{aligned}$$

Using the orthogonal matrix U and the algorithm of the QR-factorization [21, 48, and 61], we transform matrix N^T as follows:

$$\begin{bmatrix} \Lambda_k \\ 0 \end{bmatrix} = U_k N_k^T, \tag{46}$$

where Λ_k is convertible matrix.

So in accordance with (45), (46) we have

$$p_k = m_k \Lambda_k^{-1}. \tag{47}$$

Similarly we represent the right part of (46) as the product of two rectangular matrices and use the QR-factorization of these matrices, which produces the orthogonal matrix Z_k:

$$m_{k+1} m_{k+1}^T = T_k T_k^T,$$

$$T_k = [\Phi_k \quad p_k \quad q_k]; \tag{48}$$

$$\begin{bmatrix} X_k^T \\ 0 \end{bmatrix} = Z_k T^T, \tag{49}$$

$$m_{k+1} = X_k. \tag{50}$$

Thereby using given m_k, η_k, the factor p_k is computed in accordance with (46), (47), and then the factor m_{k+1} is computed in accordance with (48 – 50). Now we can withdraw from assumption about matrix R convertibility. For this purpose we can eliminate matrix η_k^{-1} and transform matrix N_k. Using orthogonal matrix Ω_k in QR – factorization, we transform matrix $m_k^T H^T$ to the following expression:

$$\begin{bmatrix} Y_k \\ 0 \end{bmatrix} = \Omega_k m_k^T H^T.$$

Assuming existence of matrix Y_k^{-1}, we introduce matrix $\tilde{W}_k = diag\left\{\eta_k^T Y_k^{-1}, E\right\}$. Using this matrix we can transform matrix N_k as follows:

$$N_k = \Omega_k^T \tilde{W}_k^{-1} \tilde{N}_k,$$

$$\tilde{N}_k = \begin{bmatrix} \tilde{W}_k \Omega_k & \begin{bmatrix} E \\ 0 \end{bmatrix} \end{bmatrix}.$$

It is obvious, that: $\left(N_k N_k^T\right)^{-1} = \Omega_k^T \tilde{W}_k^T \left(\tilde{N}_k \tilde{N}_k^T\right)^{-1} \tilde{W}_k \Omega_k$. Therefore, if orthogonal matrix $\tilde{U}$ transforms matrix $\tilde{N}^T$ likewise to (46), i.e.

$$\begin{bmatrix} \tilde{S}_k^T \\ 0 \end{bmatrix} = \tilde{U}_k \tilde{N}_k^T,$$

then it will be possible to right down the following expression for matrix p_k,which doesn't contain η_k^{-1}:

$$p_k = m_k \Omega_k^T \tilde{W}_k^T \tilde{S}_k^{-1}.$$

It is obvious, that in the case of the matrix η_k singularity relations, which define matrix m_{k+1}, will not be changed, i.e. matrix m_{k+1} will be defined by expressions (48) – (50).

5.2. Usage magnetometer and altimeter signals for SINS errors correction [47].

In previous item the process of SINS correction based on SINS/GPS fusion was described. In this item this problem will be generalized via including in the sources of external correction the magnetometer and altimeter signals.

So besides the statistical parameters of the signals and the measurements noises, the initial information for the correction algorithm operation is the discrepancy vector (ε_k), which is computed as the difference between the GPS signals (vector z) and the estimation of the current values of UAV coordinates and velocity (vector $H\ \bar{x}_k$):

$$\varepsilon_k = z_k - H\ \ \bar{x}_k. \tag{51}$$

It is natural, that considered generalization of the problem statement must be associated with the generalization of the computational procedure of the corresponding discrepancy vector estimation. Thus, in a case of augmentation of the measurement channels with the altimeter readouts, this generalization is reduced to the corresponding extension of the vector z_k and matrix H in (30). However the inclusion of the magnetometer readouts requires some additional considerations. For the sake of the computational simplicity we will consider (or simulate) the information channel, which is associated with the magnetometer, using following way. It is supposed that the vector of magnetometer readouts ($\bar{m}$) is measured in the body frame on the moving vehicle. In the Earth frame this vector is the unit vector directed along the $x - axis(m = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T)$. So, possessing values of the $\bar{m}$ vector and estimation of the $\bar{A}$-DCM, we will find the estimation of the small turn angle γ, which defines the attitude error. For this aim it is possible to use the following relation (see, for instance, expression (6.4) in [44]):

$$\bar{A}\bar{m} - m = m \times \gamma. \tag{52}$$

Expression (31) can be interpreted as the formalization of the assumption, that the small turn vector γ produces the rotation of the vector m till the coincidence with vector $\bar{A}\bar{m}$. Assume also, that vectors γ and m are mutually orthogonal vectors. Multiplying both parts of the expression (52) by $m\times$, we will receive the following expression for γ:

$$-\gamma = m \times (\bar{A}\ \hat{\bar{m}} - m). \tag{53}$$

As the consequence of the accepted assumption of the vectors m and γ orthogonality, the 1st component of the vector γ will be equal to zero and it can be excluded from consideration. Remained components of the vector γ can be interpreted as the result of the measurement of two corresponding components of the vector μ, appearing in (37).

Thus, taking into account aforementioned remarks, in a case, when the information about the magnetometer and altimeter readouts is available along with the GPS signals, it is possible to accept as the vector z_k appearing in (51), the following vector $z_k \in R^{9\times1}$:

$$z_k = \left[-\tilde{\gamma}, \tilde{v}, \tilde{r}\right]^T, \tag{54}$$

where $\tilde{\gamma} \in R^{2\times1}$ stands for vector, consisting of the last two components of the vector γ defined by (32), $\tilde{v} \in R^{3\times1}$ stands for the UAV velocity estimation obtained from GPS, and $\tilde{r} \in R^{4\times1}$ stands for the vector of the UAV coordinates, obtained with help of GPS and altimeter readouts. It is naturally, that the matrix H in (39) must be correspondingly changed.

Now we can consider more realistic case of the SINS correction via magnetometer signals. As opposed to the previous case we will withdraw from aforementioned assumption, that in the Earth frame the vector of the magnetic field is determined by the OX-axis unit vector, i.e. $m = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. We will show that in this general case it is possible to use after corresponding modification the algorithm mentioned above.

Thus, let m_τ is the unit vector, which determines the magnetic field but doesn't coincide with the OX-axis unit vector. Let the orthogonal matrix τ possesses the following property:

$$\tau m_\tau = m = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T. \tag{55}$$

In this case it is expedient to consider the first 3 components of the vector x_k, which determine the small turn vector, in the frame determined by matrix τ appearing in (55). In other words, it is necessary to introduce the small turn vector $\bar{\mu}$, which is associated with μ, in a following way:

$$\bar{\mu} = \tau\mu. \tag{56}$$

In connection with (56), the matrix Φ_k appearing in (38) must be undergone to the following linear transformation:

$$\bar{\Phi}_k = \theta\Phi_k\theta^T, \theta = diag\left\{\ \tau,\quad I,\quad I\ \right\}. \tag{57}$$

This matrix $\bar{\Phi}_k$ must be used in the relations of the item 3. Now we consider the modifications, which are necessary to apply the procedure defined by the relations (51) – (53). The analog of the relation (52) in this case is the following relation:

$$\bar{A}\bar{m}_\tau - m_\tau = m_\tau \times \gamma_\tau. \tag{58}$$

The vector $\bar{m}_\tau$ in the expression (58) is the result of the magnetic field strength vector measurement in the body frame, γ_τ is the vector corresponding to a small turn. Multiplying (58) by τ, we will obtain the following relation:

$$\tau\bar{A}\bar{m}_\tau - m = m \times \bar{\gamma}, \qquad \bar{\gamma} = \tau\gamma_\tau. \tag{59}$$

The analog of (53) is derived from (59):

$$-\bar{\gamma} = m \times \tau\bar{A}\bar{m}_\tau. \tag{60}$$

As far as the vector γ is orthogonal to the vector m, then the 1st component of the vector $\bar{\gamma}$ is zero. As it follows from (60), the vector $\tilde{\gamma}$ appearing in (54) will possess only two components, which coincide with last two components of the vector $\bar{\gamma}$. So in the considered case the vector z_k in (54) is determined.

Then it is necessary to multiply the vector x_k estimation, obtained in accordance with (37), by matrix θ^T. It is stipulated by the fact, that the first three components of the vector x_k correspond to the vector $\bar{\mu}$, which is connected by the relation (56) with the small turn vector (μ) in the initial frame.

6. Detection of the faulty sensor [46]

Note that along with given equations, when that or those uncertainty is present in the system (see, for instance, [23]), the problems of the "faulty" element detection also attracts increasing attention of researchers [20, 23, 34, 49, 53, 59]. We consider that the faulty element is the element, which changed essentially its characteristics. In this connection we consider the problem of detection of the faulty RG included in the SINS. Thus, it was assumed above, that 3 rate gyros (RG) are included in the SINS. However it might be expedient to increase the number of RG for obtaining possibility to exclude the faulty sensor and therefore to increase the reliability level of the SINS (see, for instance, [20]). Then an output of the RG – block y is connected with measured value of the angular rate ω by the next equation (analog of the relations (2), (8) in [20]):

$$y = A_R\omega + e, \tag{61}$$

where: $y \in R^{n\times1}$, $e \in R^{n\times1}$, $\omega \in R^{(n-m)\times1}$, and $A_R \in R^{n\times(n-m)}$, i.e. m is the redundant sensors amount. It is assumed that in a case of the "faulty" $i-th$ sensor, the error appearing in (61) has the following form:

$$e_{*i} = [0, \ldots, a_i, 0, \ldots, 0]^T. \tag{62}$$

The non-zero element a_i in the vector e_{*i} determined by (61) stands at i-th place. For identification of the faulty RG it is necessary to determine the value of the index i and, if it is desirable, to find the value a_i.

Applying the least square method for solution of equation (61), we can determine the value ω as follows:

$$\omega = Z(y-e), Z = (A_R^T A_R)^{-1} A_R^T. \tag{63}$$

Note, that if the vector e is determined by (63), then:

$$Ze = a_i z_i, \tag{64}$$

where z_i is the i-th column of the matrix Z, which is determined by (63).

Note also, that the algorithm of the systematic RG error compensation described in the item 3 allows obtaining estimation $\delta \overset{\frown}{c}$ of the vector δc. Having this estimation it is possible to determine vector e_{*i}, i.e. the value of the index i and estimation $\overset{\frown}{a}_i$ of the value a_i.

Let the vector δc is specified by the systematic error of the i-th RG determined by (62), i.e. $\delta c = a_i z_i$ in accordance with (64). Then, assuming the estimation $\delta \overset{\frown}{c}$ to be known, the problem of determination of the index i and the value $\overset{\frown}{a}_i$ is considered.

So let we have the estimation of $\delta \overset{\frown}{c}$. The problem is formulated as the choice of the vector z_i (and finding index i and value $\overset{\frown}{a}_i$), which could approximate $\delta \overset{\frown}{c}$ in the best way. In other words, it is necessary to determine z_i and $\overset{\frown}{a}_i$, which minimizes value of the following discrepancy:

$$\begin{aligned} Dis = \left\| \delta \overset{\frown}{c} - \overset{\frown}{a}_i z_i \right\|^2 = (\delta \overset{\frown}{c} - \overset{\frown}{a}_i z_i)(\delta \overset{\frown}{c} - \overset{\frown}{a}_i z_i)^T = \\ = \overset{\frown}{a}_i^2 z_i z_i^T - 2 \overset{\frown}{a}_i z_i \delta \overset{\frown}{c}^T + \delta \overset{\frown}{c} \delta \overset{\frown}{c}^T . \end{aligned} \tag{65}$$

In accordance with (65), the value $\overset{\frown}{a}_i$, which minimizes Dis, is determined by the following relation:

$$\overset{\frown}{a}_i = z_i \delta \overset{\frown}{c}^T \Big/ z_i z_i^T . \tag{66}$$

Thus problem of the index i choice is reduced to the choice of vector z_i ($i = 1, \ldots, n$), which minimizes Dis in (65) under condition, that $\overset{\frown}{a}_i$ is determined by (66), i.e.

$$i = \arg\min_i \left\| \delta \overset{\frown}{c} - \overset{\frown}{a}_i z_i \right\| . \tag{67}$$

Then, if necessary, the value $\overset{\frown}{a}_i$ might be determined by (66). It is useful to underline, that the algorithm, described above, allows determination not only estimation of $\overset{\frown}{a}_i$, but the estimation of the faulty RG index (i) as well. The problem of the faulty RG "isolation" and corresponding reconfiguration of the system operation algorithm requires (by our opinion) separate consideration. Example, which is given below, illustrates described above procedures of determination of the estimations $\overset{\frown}{a}_i$ and index of the faulty RG (i).

Limited volume of journal publication doesn't permit to include in this review some practical examples, which illustrate the efficiency of all aforementioned algorithms. However the reader can find them in [42 – 46].

7. Conclusion

7.1. Solution of problems of navigation systems development for low-cost small UAV requires application of low-cost hardware as well as low-cost software designated to the application in these systems. One of the possible approaches for solution of the navigation software problems in this area is adoption of some simplifying assumptions, based on the properties of dynamics and kinematics of small UAV. These include: relatively small flight distances, flight periods, air speeds etc.

7.2. Taking in account these peculiarities it is possible to use the quadratic spline approximation of the sensors signals and effective approximation of "elementary" quaternion at small sampling periods, thus to obtain the solution of the rotational and translational state maintenance in quadratures. The last feature essentially simplifies the algorithms of SINS software.

7.3. Such systems need in the external correction. This problem is solved on the basis of the linearization approach and Kalman filter application. The application of this approach is complicated due to the problem of incomplete observability of the pair of the state propagation Φ_k and observation H matrices. This difficulty can be overcome via QR-decomposition of corresponding covariance matrices. Using this approach it was possible to compensate effectively SINS errors via GPS/SINS fusion.

7.4. Further improvement of the external error correction can be implemented via combination of the magnetometer and altimeter output signals for increasing of the accuracy of the navigation problem solution. All these corrections give possibility to compensate the attitude, velocity and position errors, as well as systematic errors of rate gyros.

7.5. In cases of the fast-spinning UAV the problem of withdrawal from RG usage and creation accelerometer-based SINS (ASINS) arises. Aforementioned mathematical background allows proposing algorithms of ASINS operation with correction from GPS, having possibility of attitude determination. These algorithms are represented in review, as well as the solution of the problem of the ASINS accuracy improvement via increasing of the sensors redundancy degree.

7.6. It is shown in this review, that on the basis of the proposed software it is possible to develop algorithms for fault-tolerant navigation system in a case of redundant sensors. The algorithm of faulty sensor identification is also proposed.

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